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The 11-dimensional Metric for AdS/CFT RG Flows with Common $SU(3)$ Invariance

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Abstract

The compact 7-manifold arising in the compactification of 11-dimensional supergravity is described by the metric encoded in the vacuum expectation values(vevs) in $d = 4$, $\mathcal{N} = 8$ gauged supergravity. Especially, the space of $SU(3)$-singlet vevs contains various critical points and RG flows(domain walls) developing along AdS_4 radial coordinate. Based on the nonlinear metric ansatz of de Wit-Nicolai-Warner, we show the geometric construction of the compact 7-manifold metric and find the local frames(siebenbeins) by decoding the $SU(3)$-singlet vevs into squashing and stretching parameters of the 7-manifold. Then the 11-dimensional metric for the whole $SU(3)$-invariant sector is obtained as a warped product of an asymptotically AdS_4 space with a squashed and stretched 7-sphere. We also discuss the difference in the 7-manifold between two sectors, namely $SU(3) \times U(1)$-invariant sector and G_2-invariant sector. In spite of the difference in base 6-sphere, both sectors share the 4-sphere of $\mathbf{CP}^2$ associated with the common $SU(3)$-invariance of various 7-manifolds.

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1 Introduction

One of the crucial points of de Wit-Nicolai theory [1] is the presence of warp factor $\Delta(x, y)$. When one reduces 11-dimensional supergravity theory to four-dimensional $\mathcal{N} = 8$ gauged supergravity, the four-dimensional spacetime is warped by this factor which depends on both four-dimensional coordinate x^μ and 7-dimensional internal coordinate y^m. This warp factor provides an understanding of the different relative scales of the 11-dimensional solutions corresponding to the critical points in $\mathcal{N} = 8$ gauged supergravity. One writes down 11-dimensional metric as warped product ansatz

$$ds_{11}^2 = ds_4^2 + ds_7^2 = \frac{1}{\Delta(x,y)}\, g_{\mu\nu}(x)\, dx^\mu dx^\nu + G_{mn}(x,y)\, dy^m dy^n, \tag{1.1}$$

where $\mu, \nu = 1, 2, \cdots, 4$ and $m, n = 1, 2, \cdots, 7$. The nonlinear metric ansatz in [2] provides the explicit formula (5.1) for the 7-dimensional inverse metric $G^{mn}(x, y)$, which is encoded by the warp factor $\Delta(x, y)$, 28 Killing vectors $\overset{\circ}{K}{}^{mIJ}(y)$ on the round 7-sphere(expressed through the Killing spinors) and 28-beins $u_{ij}{}^{IJ}(x), v_{ijIJ}(x)$ in four-dimensional gauged $\mathcal{N} = 8$ supergravity: the dependence of x appears in the first and last one.

Having established the holographic duals [3, 4, 5] of both supergravity critical points and small perturbations around them, one can proceed the supergravity description of the renormalization group(RG) flow between the two fixed points. The supergravity scalars tell us what relevant operators in the dual field theory would drive a flow to the fixed point in the infrared(IR). To construct the superkink corresponding to the supergravity description of the nonconformal RG flow connecting two critical points in $d = 3$ conformal field theories, a three-dimensional Poincare invariant metric takes the form

$$g_{\mu\nu}(x)\, dx^\mu dx^\nu = e^{2A(r)}\, \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'} + dr^2, \tag{1.2}$$

where $\eta_{\mu'\nu'} = (-, +, +)$, $r = x^4$ is the coordinate transverse to the domain wall and $A(r)$ is the scale factor.

The 70 real scalars of $\mathcal{N} = 8$ supergravity parametrize [6] the coset space $E_{7(7)}/SU(8)$. The scalar potential is a function of these 70 scalars and this number is too large to be managed practically. From the possible embeddings of $SU(3)$, the 70 scalars contain 6 singlets of $SU(3)$(three from 35 scalars $\mathbf{35_v}$ plus three from 35 pseudo-scalars $\mathbf{35_c}$). But the scalar potential depends on only four real fields due to the $SO(8)$-invariance of the potential and a larger invariance of the $SU(3)$-invariant sector. The explicit construction of 28-beins $u_{ij}{}^{IJ}(x)$ and $v_{ijIJ}(x)$ in terms of these fields has been found in [7]. It is known [8] that there exist five nontrivial critical points for the scalar potential of gauged $\mathcal{N} = 8$ supergravity: $SO(7)^+, SO(7)^-, G_2, SU(4)^-$ and $SU(3) \times U(1)$. Among them G_2-invariant 7-ellipsoid and $SU(3) \times U(1)$-invariant stretched 7-ellipsoid are stable and supersymmetric.

In [9], RG flow from $\mathcal{N} = 8$, $SO(8)$-invariant ultraviolet(UV) fixed point to $\mathcal{N} = 2$, $SU(3) \times U(1)$-invariant IR fixed point was found by studying de Wit-Nicolai potential which is invariant under $SU(3) \times U(1)$ group. For this interpretation it was crucial to know the form of superpotential that was encoded in the structure of T-tensor of a theory. Moreover, one can proceed this direction for $\mathcal{N} = 1$, G_2-invariant fixed point [7, 10](See also [11]). It turned out that we found first-order BPS equations, namely holographic RG flow equations, by recognizing some algebraic(and essential) relation between the superpotential and derivative of it with respect to field. Their solutions constitute supersymmetric domain walls *both* from direct minimization of energy-functional and from supersymmetry transformation rules.

The M-theory lift of a supersymmetric RG flow is achieved as follows. First we impose the nontrivial r-dependence of vacuum expectation values(vevs) subject to the four-dimensional RG flow equations. Then the geometric parameters in the 7-manifold metric at certain critical point are controlled by the RG flow equations so that they can be smoothly extrapolated from the critical point. Secondly we make an appropriate ansatz for the 11-dimensional three-form gauge field. If the ansatz is correct, the 11-dimensional Einstein-Maxwell(bosonic) equations can be finally solved by using the RG flow equations to complete the M-theory lift. Based on this prescription, an exact solution to the 11-dimensional bosonic equations corresponding to the M-theory lift of the $\mathcal{N} = 2$, $SU(3) \times U(1)$-invariant RG flow was found in [12]. Its Kähler structure was extensively studied in [13]. Similarly, the M-theory lift of the $\mathcal{N} = 1$, G_2-invariant RG flow was done in [10].

In AdS/CFT context [3, 4, 5], the above two membrane flows are holographic dual of flows of the maximally supersymmetric $\mathcal{N} = 8$ scalar-fermion theory in three-dimensions. It is still unclear how they are related both in the bulk supergravity and in the boundary field theory. In order to answer this question, we would like to achieve the M-theory lift of *whole* $SU(3)$-invariant sector, including the above five critical points and RG flows in $d = 4$, $\mathcal{N} = 8$ gauged supergravity. As the first step toward this goal, we need to know the complete metric (1.1) together with (1.2). To solve the 11-dimensional bosonic equations by utilizing the RG flow equations in [7], we have to further make an appropriate ansatz for the 11-dimensional three-form gauge field. It will be a natural extension of the Freund-Rubin parametrization [14] but will be more complicated due to its nontrivial AdS_4 radial coordinate dependence. Even though the nonlinear metric ansatz in [2] provides the the 11-dimensional four-form gauge field strengths also,[1] they are encoded in $SU(3)$-singlet vevs in a complicated way. In this paper, we concentrate on the 11-dimensional metric for the $SU(3)$-invariant sector and postpone the field strength ansatz as well as solving the 11-dimensional bosonic equations to future work.

We begin our analysis in section 2 by summarizing relevant aspects of the 11-dimensional

[1] They are given by Eq. (7.6) in [15].

metric for $\mathcal{N} = 2$, $SU(3) \times U(1)$-invariant membrane flow found in [12]. In section 3, we review the 11-dimensional metric for $\mathcal{N} = 1$, G_2-invariant membrane flow found in [10]. In section 4, we summarize the four-dimensional RG flow equations [7], namely the first-order BPS equations for the $SU(3)$-invariant sector in $d = 4$, $\mathcal{N} = 8$ gauged supergravity. In section 5, that is the main content of this paper, we show the geometric construction of the compact 7-manifold metric and find the local frames(siebenbeins) by decoding the $SU(3)$-singlet vevs into squashing and stretching parameters of the 7-manifold. We mainly use Hopf fibration on $\mathbf{CP}^3$ as global 7-dimensional coordinates. In section 6, we summarize our results and will discuss about future direction. In appendix A, we list all the components of the 7-dimensional inverse metric generated from the metric formula (5.1) by using $\mathbf{R}^8$ embedding. In appendix B, some preliminaries of Hopf fibration are presented. In appendix C, we show the other set of global 7-dimensional coordinates useful to describe the G_2-invariant critical point.

2 The 11-dimensional metric for $\mathcal{N} = 2, SU(3) \times U(1)$-invariant membrane flow

The 28-beins (u, v) or two vevs (ρ, χ) [2] are given by functions of the AdS_4 radial coordinate $r = x^4$. The metric formula (5.1) generates the 7-dimensional metric from the two input data of AdS_4 vevs (ρ, χ). The $SU(3) \times U(1)$-invariant RG flow subject to the first order differential equations on two vevs [9] is a trajectory in (ρ, χ)-plane and is parametrized by the AdS_4 radial coordinate.

Let us introduce the standard metric corresponding to an ellipsoidally squashed 7-sphere with a stretched Hopf fiber. The stretching factor is characterized by the vev χ and $\chi = 0$ corresponds to the round 7-sphere. Then the isometry $SO(8)$ of 7-sphere breaks into $U(4)$. The metric is also ellipsoidally squashed and it reduces the broken isometry to $SU(3) \times U(1) \times U(1)$. Let us introduce a diagonal 8×8 matrix

$$Q_{AB} = \text{diag}\left(\rho(r)^{-2}, \ldots, \rho(r)^{-2}, \rho(r)^6, \rho(r)^6\right).$$

Recall that A_1 tensor of this theory has two distinct eigenvalues with degeneracies 6, 2. This behavior reflects also in the deformation matrix here. The metric on the deformed $\mathbf{R}^8$(Cartesian coordinates on $\mathbf{R}^8$ are denoted by X^A with $\sum_{A=1}(X^A)^2 = 1$) can be written as(See also [13])

$$ds^2(\rho(r), \chi(r)) = dX^A Q(r)^{-1}_{AB}\, dX^B + \frac{\sinh^2 \chi(r)}{\xi(r, \mu)^2} \left(X^A J_{AB}\, dX^B\right)^2 \tag{2.1}$$

[2]The variables λ and λ' in [9] with $\alpha = 0, \phi = \pi/2$ are related to $\lambda = 4\sqrt{2} \ln \rho$, $\lambda' = \sqrt{2}\chi$. In terms of a, b, c and d we will define later, $\rho = a^{1/4} = b^{-1/4}$ and $\chi = \cosh^{-1} c = \cosh^{-1} d$.

where the quadratic form $\xi(r,\mu)^2 \equiv X^A Q_{AB} X^B$ is now given by using the parametrization of [12]

$$\xi(r,\mu)^2 = \rho(r)^{-2}\cos^2\mu + \rho(r)^6 \sin^2\mu$$

where μ is one of the 7-dimensional internal coordinates and $\xi(r,\mu)^2$ becomes 1 when $\rho = 1$. For $\rho = 1, \chi = 0$, it provides the trivial vacuum of $SO(8)$ maximal supersymmetric critical point. The antisymmetric Kähler form J_{AB} has nonzero elements $J_{12} = J_{34} = J_{56} = J_{78} = 1$.

Applying the Killing vector together with the 28-beins (u, v) to the metric formula (5.1), we obtain a "raw" inverse metric $\Delta(x,y)^{-1}G^{mn}(x,y)$ including the warp factor $\Delta(x,y)$ not yet determined. Substitution of this raw inverse metric into the definition of warp factor

$$\Delta(x,y)^{-1} \equiv \sqrt{\det(G^{mn}(x,y)\,\overset{\circ}{g}_{np}(y))}, \tag{2.2}$$

where $\overset{\circ}{g}_{np}(y)$ is a metric of the round 7-sphere, will provide a self-consistent equation for $\Delta(x,y)$. For the $SU(3)\times U(1)$-invariant RG flow, solving this equation gives rise to the warp factor

$$\Delta(r,\mu) = (\xi(r,\mu)\cosh\chi(r))^{-\frac{4}{3}}\,. \tag{2.3}$$

Then we substitute this warp factor into the "raw" inverse metric and obtain the 7-dimensional metric:

$$ds_7^2 = G_{mn}(x,y)\,dy^m dy^n = \sqrt{\Delta(r,\mu)}L^2 ds^2(\rho(r),\chi(r)) \tag{2.4}$$

together with (2.1) and (2.3) where L is a radius of round 7-sphere. The metric (2.1) now is warped by a factor $\sqrt{\Delta(r,\mu)}$. The nonlinear metric ansatz combines the 7-dimensional metric with the four-dimensional metric with warp factor to yield the 11-dimensional warped metric characterized by (1.1), (1.2), (2.4), (2.3) and (2.1).

3 The 11-dimensional metric for $\mathcal{N}=1, G_2$-invariant membrane flow

The two vevs (λ, α) [3] are given by functions of the AdS_4 radial coordinate r. The metric formula (5.1) generates the 7-dimensional metric from the two input data of AdS_4 vevs (λ, α). The G_2-invariant RG flow subject to the first order differential equations on these vevs [7, 10] is a trajectory in (λ,α)-plane and is parametrized by the AdS_4 radial coordinate. We will use (a, b) defined by

$$\begin{aligned} a(r) &\equiv \cosh\left(\frac{\lambda(r)}{\sqrt{2}}\right) + \cos\alpha(r)\,\sinh\left(\frac{\lambda(r)}{\sqrt{2}}\right), \\ b(r) &\equiv \cosh\left(\frac{\lambda(r)}{\sqrt{2}}\right) - \cos\alpha(r)\,\sinh\left(\frac{\lambda(r)}{\sqrt{2}}\right). \end{aligned} \tag{3.1}$$

[3] One can obtain these vevs from $SU(3)$-invariant sector by restricting four arbitrary fields $\lambda, \lambda', \alpha, \phi$ to $\lambda' = \lambda$ and $\phi = \alpha$. This is equivalent to $c = a$ and $d = b$ where a, b, c and d are defined as the one in Section 5.

Let us introduce the standard metric of a 7-dimensional ellipsoid. Using the diagonal 8×8 matrix Q_{AB} given by [4]

$$Q_{AB} = \mathrm{diag}\left(b(r)^2, \ldots, b(r)^2, a(r)^2\right), \tag{3.2}$$

the metric of a 7-dimensional ellipsoid can be written as

$$ds^2_{EL(7)}(a(r), b(r)) = dX^A Q(r)^{-1}_{AB} \, dX^B. \tag{3.3}$$

This can be rewritten in terms of the 7-dimensional coordinates y^m(See [10] for the explicit relations between $\mathbf{R}^8$ coordinates X^A and y^m) such that

$$ds^2_{EL(7)}(a(r), b(r)) = b(r)^{-2} \left[a(r)^{-2} \, \xi(r,\theta)^2 \, d\theta^2 + \sin^2\theta \, d\Omega_6^2 \right], \tag{3.4}$$

where $\theta = y^7$ must be identified with the fifth coordinate in 11 dimensions and the quadratic form $\xi(r,\theta)^2$ is given by

$$\xi(r,\theta)^2 = a(r)^2 \cos^2\theta + b(r)^2 \sin^2\theta \tag{3.5}$$

which becomes 1 for $a = b = 1$ corresponding to $SO(8)$ trivial vacuum.

Note that the geometric parameters (a, b) for the 7-ellipsoid can be identified with the two vevs (a, b) defined in (3.1). This was the reason why we prefer the vevs (a, b) rather than (λ, α). Applying the Killing vector together with the 28-beins to the metric formula (5.1) like we did before, we obtain a "raw" inverse metric $\Delta(x,y)^{-1} G^{mn}(x,y)$ including the warp factor $\Delta(x,y)$. For the G_2-invariant RG flow, solving the condition yields the warp factor

$$\Delta(r,\theta) = a(r)^{-1} \, \xi(r,\theta)^{-\frac{4}{3}}. \tag{3.6}$$

Then we substitute this warp factor into the raw inverse metric and obtain the 7-dimensional metric as follows:

$$ds_7^2 = G_{mn}(x,y) \, dy^m dy^n = \sqrt{\Delta(r,\theta) \, a(r)} \; b(r)^2 L^2 ds^2_{EL(7)}(a(r), b(r)). \tag{3.7}$$

The 7-dimensional metric (3.3) is warped by a factor $\sqrt{\Delta(r,\theta) \, a(r)}$. The nonlinear metric ansatz with the warp factor yields the warped 11-dimensional metric described by (1.1), (1.2), (3.7), (3.3) and (3.6).

[4]Note that the diagonal matrix here is different from the one in [10] and is defined as the old one multiplied by $b(r)^2$.

4 Holographic RG flow for $SU(3)$-invariant sector

It is known [8] that $SU(3)$-singlet space with a breaking of the $SO(8)$ gauge group into a group which contains $SU(3)$ may be written as four real parameters $\lambda(r), \lambda'(r), \alpha(r)$ and $\phi(r)$. The vacuum expectation value of 56-bein for the $SU(3)$-singlet space, that is an invariant subspace under a particular $SU(3)$ subgroup of $SO(8)$, can be parametrized by

$$\phi_{ijkl} = \lambda(r)\,\cos\alpha(r)\,Y^{1\,+}_{ijkl} + \lambda(r)\,\sin\alpha(r)\,Y^{1\,-}_{ijkl} + \lambda'(r)\,\cos\phi(r)\;\;Y^{2\,+}_{ijkl} + \lambda'(r)\,\sin\phi(r)\,Y^{2\,-}_{ijkl}\;,$$

where the scalar and pseudo-scalar singlets of $SU(3)$ are given by

$$\begin{aligned} Y^{1\,\pm}_{ijkl} &= \varepsilon_{\pm}\Big[\,(\delta^{1234}_{ijkl} \pm \delta^{5678}_{ijkl}) + (\delta^{1256}_{ijkl} \pm \delta^{3478}_{ijkl}) + (\delta^{3456}_{ijkl} \pm \delta^{1278}_{ijkl})\,\Big]\,, \\ Y^{2\,\pm}_{ijkl} &= \varepsilon_{\pm}\Big[-(\delta^{1357}_{ijkl} \pm \delta^{2468}_{ijkl}) + (\delta^{2457}_{ijkl} \pm \delta^{1368}_{ijkl}) + (\delta^{2367}_{ijkl} \pm \delta^{1458}_{ijkl}) + (\delta^{1467}_{ijkl} \pm \delta^{2358}_{ijkl})\Big]\,. \end{aligned}$$

Here $\varepsilon_+ = 1$ and $\varepsilon_- = i$ and $+$ gives the scalars and $-$ gives the pseudo-scalars of $\mathcal{N} = 8$ supergravity. The four scalars $\lambda(r), \lambda'(r), \alpha(r)$ and $\phi(r)$ in the $SU(3)$-singlet vevs parametrize an $SU(3)$-invariant subspace of the complete scalar manifold $E_{7(7)}/SU(8)$. The 56-bein $\mathcal{V}(x)$ preserving the $SU(3)$-singlet space is a 56×56 matrix whose elements are some functions of four fields $\lambda(r), \lambda'(r), \alpha(r)$ and $\phi(r)$ obtained by exponentiating the above vacuum expectation value ϕ_{ijkl}. Then the 28-beins, u and v in terms of these fields, can be obtained as the 28×28 matrices given in the appendix A of [7].

It turned out [7] that A_1 tensor has three distinct complex eigenvalues, z_1, z_2 and z_3 with degeneracies 6, 1, and 1 respectively and has the following form

$$A_1^{IJ} = \mathrm{diag}\,(z_1, z_1, z_1, z_1, z_1, z_1, z_2, z_3)\,,$$

where the eigenvalues are some functions of $\lambda(r), \lambda'(r), \alpha(r)$ and $\phi(r)$. In particular,

$$z_3(\lambda, \lambda', \alpha, \phi) = 6e^{i(\alpha+2\phi)}p^2qr^2t^2 + 6e^{2i(\alpha+\phi)}pq^2r^2t^2 + p^3\left(r^4 + e^{4i\phi}t^4\right) + e^{3i\alpha}q^3\left(r^4 + e^{4i\phi}t^4\right),$$

and we denote hyperbolic functions of $\lambda(r)$ and $\lambda'(r)$ by the following quantities for simplicity

$$p \equiv \cosh\left(\frac{\lambda(r)}{2\sqrt{2}}\right), \quad q \equiv \sinh\left(\frac{\lambda(r)}{2\sqrt{2}}\right), \quad r \equiv \cosh\left(\frac{\lambda'(r)}{2\sqrt{2}}\right), \quad t \equiv \sinh\left(\frac{\lambda'(r)}{2\sqrt{2}}\right).$$

We refer to [7] for explicit expressions of z_1 and z_2.

The superpotential is one of the eigenvalues of A_1 tensor and the supergravity potential[5]

[5] The scalar potential can be written, by combining all the components of A_1, A_2 tensors, as

$$\begin{aligned} V(\lambda, \lambda', \alpha, \phi) &= \frac{1}{2}g^2\left(s'^4\left[(x^2+3)c^3 + 4x^2v^3s^3 - 3v(x^2-1)s^3 + 12xv^2cs^2 - 6(x-1)cs^2 + 6(x+1)c^2sv\right]\right. \\ &\quad \left. +2s'^2\left[2(c^3+v^3s^3) + 3(x+1)vs^3 + 6xv^2cs^2 - 3(x-1)cs^2 - 6c\right] - 12c\right), \end{aligned}$$

where we introduce the following quantities: $c \equiv \cosh\left(\frac{\lambda}{\sqrt{2}}\right)$, $s \equiv \sinh\left(\frac{\lambda}{\sqrt{2}}\right)$, $c' \equiv \cosh\left(\frac{\lambda'}{\sqrt{2}}\right)$, $s' \equiv \sinh\left(\frac{\lambda'}{\sqrt{2}}\right)$, $v \equiv \cos\alpha$, and $x \equiv \cos 2\phi$.

can be written in terms of superpotential as follows

$$
\begin{aligned}
W(\lambda,\lambda',\alpha,\phi) &= |z_3|, \\
V(\lambda,\lambda',\alpha,\phi) &= g^2\left[\frac{16}{3}\left(\partial_\lambda W\right)^2+\frac{2}{3p^2q^2}\left(\partial_\alpha W\right)^2+4\left(\partial_{\lambda'}W\right)^2+\frac{1}{2r^2t^2}\left(\partial_\phi W\right)^2-6W^2\right]
\end{aligned} \tag{4.1}
$$

The flow equations [7] we are interested in are

$$
\begin{aligned}
\partial_r\lambda(r) &= -\frac{8\sqrt{2}}{3}\,g\,\partial_\lambda W(\lambda,\lambda',\alpha,\phi), \\
\partial_r\lambda'(r) &= -2\sqrt{2}\,g\,\partial_{\lambda'}W(\lambda,\lambda',\alpha,\phi), \\
\partial_r\alpha(r) &= -\frac{\sqrt{2}}{3p^2q^2}\,g\,\partial_\alpha W(\lambda,\lambda',\alpha,\phi), \\
\partial_r\phi(r) &= -\frac{\sqrt{2}}{4r^2t^2}\,g\,\partial_\phi W(\lambda,\lambda',\alpha,\phi), \\
\partial_r A(r) &= \sqrt{2}\,g\,W(\lambda,\lambda',\alpha,\phi).
\end{aligned} \tag{4.2}
$$

There exist two supersymmetric critical points of both a scalar potential and a superpotential: $\mathcal{N}=1$ supersymmetric critical point with G_2-symmetry and $\mathcal{N}=2$ supersymmetric one with $SU(3)\times U(1)$-symmetry. Also there are three nonsupersymmetric critical points with $SO(7)^+, SO(7)^-$ and $SU(4)^-$-symmetries.

5 The 11-dimensional metric for AdS_4 RG flows with common $SU(3)$ invariance

In this section, we will give an ansatz for the generic $SU(3)$-invariant metric in terms of squashing deformation and the Kähler form J by using the $\mathbf{R}^8$ vector $X = Rx$. Then we invert the metric and change the $\mathbf{R}^8$ basis from X to x to compare the ansatz with the inverse metric generated by the de Wit-Nicolai-Warner formula [2, 15].

5.1 The compact 7-manifold metric encoded in data of $d=4$, $\mathcal{N}=8$ gauged supergravity

The consistency under the Kaluza-Klein compactification of 11-dimensional supergravity, or M-theory, requires that the 11-dimensional metric for RG flows with common $SU(3)$ invariance is *not* simply a metric of product space *but* Eq. (1.1) for the warped product of an asymptotically AdS_4 space, or a domain wall, with a compact 7-dimensional manifold. Moreover, the 7-dimensional space becomes a warped, squashed and stretched $\mathbf{S}^7$ and its metric is uniquely determined through the nonlinear metric ansatz developed in [2, 15]. The warped 7-dimensional inverse metric is given by

$$G^{mn}(x,y) = \frac{1}{2}\,\Delta(x,y)\left[\overset{\circ}{K}{}^{mIJ}\overset{\circ}{K}{}^{nKL} + (m \leftrightarrow n)\right]\left(u_{ij}{}^{IJ}(x) + v_{ijIJ}(x)\right)\left(\overline{u}^{ij}{}_{KL}(x) + \overline{v}^{ijKL}(x)\right) \tag{5.1}$$

where $\overset{\circ}{K}{}^{mIJ}$ denotes the Killing vector on the round $\mathbf{S}^7$ with 7-dimensional coordinate indices $m, n = 5, \ldots, 11$ as well as $SO(8)$ vector indices $I, J = 1, \ldots, 8$. The $u_{ij}{}^{IJ}$ and v_{ijIJ} are 28-beins in 4-dimensional gauged supergravity and are parametrized by the AdS_4 vacuum expectation values(vevs), λ, $\lambda^{'}$, α and ϕ, associated with the spontaneous compactification of 11-dimensional supergravity.

The 28-beins (u, v) or four vevs $(\lambda, \lambda', \alpha, \phi)$ are given by functions of the AdS_4 radial coordinate $r = x^4$. The metric formula (5.1) generates the 7-dimensional metric from the four input data of AdS_4 vevs $(\lambda, \lambda', \alpha, \phi)$. The RG flow subject to Eq. (4.2) is a trajectory in the $SU(3)$-singlet space spanned by $(\lambda, \lambda', \alpha, \phi)$ and is parametrized by the AdS_4 radial coordinate. Hereafter, instead of $(\lambda, \lambda', \alpha, \phi)$, we will use (a, b, c, d) defined by

$$\begin{aligned}
a(r) &\equiv \cosh\left(\frac{\lambda(r)}{\sqrt{2}}\right) + \cos\alpha(r)\,\sinh\left(\frac{\lambda(r)}{\sqrt{2}}\right), \\
b(r) &\equiv \cosh\left(\frac{\lambda(r)}{\sqrt{2}}\right) - \cos\alpha(r)\,\sinh\left(\frac{\lambda(r)}{\sqrt{2}}\right), \\
c(r) &\equiv \cosh\left(\frac{\lambda'(r)}{\sqrt{2}}\right) + \cos\phi(r)\,\sinh\left(\frac{\lambda'(r)}{\sqrt{2}}\right), \\
d(r) &\equiv \cosh\left(\frac{\lambda'(r)}{\sqrt{2}}\right) - \cos\phi(r)\,\sinh\left(\frac{\lambda'(r)}{\sqrt{2}}\right).
\end{aligned}$$

The inverse metric for the 7-manifold generated by the de Wit-Nicolai-Warner(dWNW) formula (5.1) is encoded in the data of 28-beins, namely a, b, c, d given above. Therefore, to get the metric available for practical purposes, we have to decode the 28-beins into the deformation parameters of the compact 7-manifold. This can be done by comparing the encoded inverse metric with the inverse metric given by some appropriate ansatz. Since we do not know what are the proper coordinates describing the compact 7-manifold yet, we will consider the $\mathbf{R}^8$ embedding by using the Cartesian coordinates on $\mathbf{R}^8$, namely $x^A, A = 1, \cdots, 8$. Now the Killing vectors defined on the round $\mathbf{S}^7$ with radius L are given by

$$\overset{\circ}{K}{}_A{}^{IJ} = L\,(\Gamma^{IJ})_{BC}(x^B \partial_A x^C - x^C \partial_A x^B) = L\left[x^B (\Gamma^{IJ})_{BA} - x^C (\Gamma^{IJ})_{AC}\right]$$

where the $\mathbf{R}^8$ coordinates x^A's are constrained on the unit round $\mathbf{S}^7$, $\sum_{A=1}^{8}(x^A)^2 = 1$, and Γ^{IJ} are the $SO(8)$ generators given in [16, 10]. Substituting this into the formula (5.1) generates the inverse metric divided by the warp factor, namely $\Delta^{-1}G^{AB}$, described by the $\mathbf{R}^8$ coordinates x^A's. We list all of its components in appendix A. Our goal is to reproduce the same inverse metric via purely geometric construction and finally to determine the metric and the warp factor separately.

The complication coming from using the $\mathbf{R}^8$ coordinates is that the Kähler form J defined by $J^2 = -I$ is not standard form in the coordinates x^A's. By taking the $SU(4)^-$-invariant limit $a = b = 1$, $d = c$ [17] in the raw inverse metric in appendix A, we obtain

$$\Delta^{-1} G^{AB} = c^2(\delta^{AB} - x^A x^B) + (1 - c^2)\,(\tilde{J}_{AC}\,x^C)(\tilde{J}_{BD}\,x^D)$$

from which one can read the Kähler form $\tilde{J}$ as an 8×8 matrix given by Eq. (A.1) in appendix A. Therefore to get the standard Kähler form J given by

$$J_{12} = J_{34} = J_{56} = J_{78} = 1, \tag{5.2}$$

we must transform the $\mathbf{R}^8$ vector x to another one $X = R\,x$ by using the 8×8 orthogonal matrix R given by Eq. (A.2) in appendix A.

5.2 Geometric construction of the compact 7-manifold metric

Geometric implication of $\mathcal{N} = 2$, $SU(3) \times U(1)$-invariant metric is as follows. First, turning off the ellipsoidal deformation, the metric of compact 7-manifold is given by Hopf fibration on $\mathbf{CP}^3$ with a stretched Hopf fiber. Fubini-Study metric on $\mathbf{CP}^3$ has $SU(4)$ invariance as can be seen from $\mathbf{CP}^3 \equiv SU(4)/(SU(3) \times U(1))$. Then turning on the ellipsoidal deformation breaks the $SU(4)$ invariance of $\mathbf{CP}^3$ down to its $SU(3) \times U(1)$ subgroup preserving the $SU(3)$ invariance of Fubini-Study metric on $\mathbf{CP}^2 \subset \mathbf{CP}^3$ as well as the $U(1)$ symmetry along the stretched Hopf fiber. Therefore to get the $SU(3)$-invariant metric, one can further break the $U(1)$ symmetry as long as the Fubini-Study metric on $\mathbf{CP}^2$ is preserved. Since the $\mathbf{S}^5$ given by Hopf fibration on $\mathbf{CP}^2$ is embedded in $\mathbf{R}^6$ spanned by $X^1, \cdots, X^6$, the first six diagonal components of the deformation matrix Q must be the same. This is the same as in $SU(3) \times U(1)$-invariant case. Recall that the $\mathbf{S}^1$ of $U(1)$ Hopf fiber on $\mathbf{CP}^3$ is embedded in $\mathbf{R}^2$ spanned by X^7, X^8. Now one can break this $U(1)$ symmetry by choosing different scaling factors in the last two diagonal components of Q. One can also decompose the Hopf fiber $(X, JdX) \equiv X^A J_{AB}\,dX^B$ into $SU(3)$-invariant pieces.

Thus we are led to the ansatz for the $SU(3)$-invariant unwarped metric:

$$ds_0^2 = (dX, Q^{-1}dX) + \frac{\gamma}{\xi^2}\Big[(U, JdU) + \zeta_1(V_1, JdV_2) + \zeta_2(V_2, JdV_1)\Big]^2 \tag{5.3}$$

where the deformation matrix Q is given by

$$Q = \mathrm{diag}(\eta, \cdots, \eta, \eta_1, \eta_2),$$

so that $\xi^2 \equiv (X, QX)$ can be the $SU(3)$-invariant norm on the 7-sphere. The $\mathbf{R}^8$ coordinates X are restricted on the round $\mathbf{S}^7$, $\sum_{A=1}^{8}(X^A)^2 = 1$, and subject to the $SU(3)$-invariant decomposition $X = U + V_1 + V_2$ with

$$U = (X^1, \cdots, X^6, 0, 0), \quad V_1 = (0, \cdots, 0, X^7, 0), \quad V_2 = (0, \cdots, 0, 0, X^8).$$

By introducing two more parameters ζ_1, ζ_2, the Hopf fiber (X, JdX) is decomposed into the $SU(3)$-invariant pieces (U, JdU), (V_1, JdV_2), (V_2, JdV_1). Now the $U(1)$ symmetry of (X, JdX) is fully broken unless $\zeta_1 = \zeta_2 = 1$.

From the ansatz (5.3), one can read the matrix g as follows.

$$g = Q^{-1} - \frac{1}{\xi^2} X X^T + \frac{\gamma}{\xi^2} F \tag{5.4}$$

where we have defined F as

$$F = (JU + \zeta_1 JV_1 + \zeta_2 JV_2)\,(JU + \zeta_1 JV_1 + \zeta_2 JV_2)^T. \tag{5.5}$$

We notice that the first two terms in g combine into the projection operator for the direction transverse to QX. This must be the case for the whole of g since the metric g describes the embedding of the 7-sphere into $\mathbf{R}^8$ by restricting the $\mathbf{R}^8$ into the subspace transverse to QX. Thus we require that F must project out the deformed vector QX, namely $FQX = 0$ which restricts ζ_1, ζ_2 to be $\zeta_2\eta_1 = \zeta_1\eta_2$ and makes $(FQ)^2$ become proportional to FQ. We further require that

$$(FQ)^2 = \xi^2 FQ, \tag{5.6}$$

which can be achieved by choosing

$$\zeta_1 = \frac{1}{\zeta_2} = \sqrt{\frac{\eta_1}{\eta_2}}.$$

The deformation parameters η, η_1, η_2, γ must be determined by comparing the inverse of g with the inverse metric generated by dWNW formula. However, the 8×8 matrix g is a projection operator describing the $\mathbf{R}^8$ embedding of the compact 7-manifold and is rank 7. It does not have its inverse in ordinary sense. Therefore we have to define the inverse of g, say g^{-1}, as an 8×8 matrix satisfying

$$g^{-1} g\, g^{-1} = g^{-1} \;\longleftrightarrow\; g\, g^{-1} g = g. \tag{5.7}$$

By looking at the $SU(3)$-invariant norm $\xi^2 = (X, QX)$, we realize that QX is a vector dual to X. This implies that g^{-1} must be a projection operator transverse to X as g is so for QX. Now we suppose that g^{-1} is given by

$$g^{-1} = Q - \frac{1}{\xi^2}\,(QX)(QX)^T - \left(\frac{\gamma}{\gamma + 1}\right) \frac{1}{\xi^2}\, QFQ. \tag{5.8}$$

which in fact projects out X by using $FQX = 0$. One can see by using Eq. (5.6) that g, g^{-1} in Eqs. (5.4), (5.8) indeed satisfy the inversion relation (5.7).

Thus the inverse metric g^{-1} in Eq. (5.8) can be compared with the warped inverse metric G^{-1} obtained by dWNW formula. They are related via

$$\Delta^{-1}G^{-1} = \xi^2 N R^T g^{-1} R, \tag{5.9}$$

where ξ^2 in the right hand side is necessary to get the left hand side just as a polynomial of the $\mathbf{R}^8$ coordinates x^A's as in appendix A. The normalization factor N, as well as the deformation parameters η, η_1, η_2 and γ, will be determined as a function of a, b, c, d. We also have to replace the $\mathbf{R}^8$ vector X with $R\,x$ in the right hand side. Substituting the results in appendix A into the left hand side, one can get

$$\eta_1 = \frac{ac}{bd}\,\eta, \quad \eta_2 = \frac{ad}{bc}\,\eta, \quad N = \frac{bcd}{\eta^2}, \quad \gamma = \frac{cd}{ab} - 1.$$

where η can be fixed by writing it like as $\eta = a^{m_1} b^{m_2} c^{m_3} d^{m_4}$ and using the two limit values: $SU(3)\times U(1)$-invariant case and G_2-invariant case. For the former, we have $b = 1/a = \rho^{-4}$ and $d = c = \cosh\chi$. In this case, η was given as ρ^{-2}. Therefore we should have $m_3 = -m_4$ and $m_1 - m_2 = -1/2$. For the latter, one has $c = a$ and $d = b$ and η was given as b^2. Therefore $m_1 + m_3 = 0$ and $m_2 + m_4 = 2$. The exponents of m_1, m_2, m_3 and m_4 are uniquely determined to yield

$$\eta = \frac{a^{\frac{3}{4}} b^{\frac{5}{4}} d^{\frac{3}{4}}}{c^{\frac{3}{4}}}, \quad \eta_1 = \frac{a^{\frac{7}{4}} b^{\frac{1}{4}} c^{\frac{1}{4}}}{d^{\frac{1}{4}}}, \quad \eta_2 = \frac{a^{\frac{7}{4}} b^{\frac{1}{4}} d^{\frac{7}{4}}}{c^{\frac{7}{4}}}, \quad N = \frac{c^{\frac{5}{2}}}{a^{\frac{3}{2}} b^{\frac{3}{2}} d^{\frac{1}{2}}} L^{-2}. \tag{5.10}$$

Here a comment on G_2-invariant limit must be in order. In the limit, we have $c = a$ and $d = b$ so that Eq. (5.10) provides $\eta = \eta_2 = b^2$ and $\eta_1 = a^2$. Hence we get $Q = \mathrm{diag}\,(b^2, \ldots, b^2, a^2, b^2)$ which is different from Q in section 3 (See Eq. (3.2)) by the interchange between the seventh and the eighth components. To fix this mismatch, we only have to interchange X^7 with X^8 by an orthogonal transformation. However, if we do so the Kähler form J is also transformed and is no longer the standard form given in Eq. (5.2). Nevertheless, this does not cause any problem in G_2-invariant case as well as in $SO(7)^{\pm}$-invariant cases since the 7-dimensional metric has no contribution from the Kähler form J.

5.3 The local frames for the compact 7-manifold

Now that we have decoded the encoded output of dWNW formula given in appendix A by determining all the geometric parameters in the $SU(3)$-invariant ansatz (5.3). They are given by Eq. (5.10). However, the dWNW formula generates the involved inverse metric $\Delta^{-1}G^{-1}$ without giving the warp factor Δ separately. Therefore to get the full 7-dimensional metric, we have to separate out the warp factor from the obtained results. Recall that the inverse metric

g^{-1} is related to $\Delta^{-1}G^{-1}$ via Eq. (5.9). Inverting Eq. (5.9) provides the warped 7-dimensional metric G involving the warp factor. Since g^{-1} is the inverse of g in Eq. (5.4), G is given in the $\mathbf{R}^8$ basis X by

$$G = \frac{1}{\Delta\,\xi^2 N}\,g \equiv \frac{L^2}{\xi^2\Delta}\left(\frac{a^{\frac{3}{2}}b^{\frac{3}{2}}d^{\frac{1}{2}}}{c^{\frac{5}{2}}}\right)\left[Q^{-1} - \frac{1}{\xi^2}\,XX^T + \frac{\gamma}{\xi^2}\,F\right]. \tag{5.11}$$

To determine the warp factor Δ, the easiest way is to find out the 7-dimensional local frames (or siebenbeins) E^i $(i = 1, \ldots, 7)$ defined as

$$ds_7^2 \equiv (dX, G\,dX) = L^2 \sum_{i=1}^{7} E^i \otimes E^i.$$

By using the wedge product of E^i's, the defining equation (2.2) of the warp factor Δ can be written as

$$\Delta \equiv (\Omega_7)^{-1} \bigwedge_{i=1}^{7} E^i, \tag{5.12}$$

where Ω_7 is the volume element of the unit round $\mathbf{S}^7$. For convenience, let us introduce the unwarped local frames e^i $(i = 1, \cdots, 7)$ defined by

$$ds_0^2 \equiv (dX, g\,dX) = \sum_{i=1}^{7} e^i \otimes e^i, \tag{5.13}$$

factorizing the Δ dependence of E^i's. Eq. (5.12) then turns out to be the self-consistent equation for Δ:

$$\Delta = \left[\frac{1}{\xi\Delta^{\frac{1}{2}}}\left(\frac{a^{\frac{3}{4}}b^{\frac{3}{4}}d^{\frac{1}{4}}}{c^{\frac{5}{4}}}\right)\right]^7 (\Omega_7)^{-1} \bigwedge_{i=1}^{7} e^i \tag{5.14}$$

where the wedge product of e^i's is calculable without knowing the warp factor. Hence in order to determine the warp factor Δ, we only have to find out the unwarped frames e^i's and to calculate the wedge product of them.

In terms of deformation parameters and the Kähler form J, the line element ds_0^2 in Eq. (5.13) is written explicitly as (See Eq. (5.3))

$$\begin{aligned} ds_0^2 &= \frac{1}{\eta}\,(dU)^2 + \frac{1}{\eta_1}\,(dV_1)^2 + \frac{1}{\eta_2}\,(dV_2)^2 \\ &\quad + \frac{\gamma}{\xi^2}\left[(U, JdU) + \sqrt{\frac{\eta_1}{\eta_2}}\,(V_1, JdV_2) + \sqrt{\frac{\eta_2}{\eta_1}}\,(V_2, JdV_1)\right]^2, \end{aligned} \tag{5.15}$$

with $\gamma = cd/ab - 1$. We will use the same 7-dimensional coordinatization of U, V_1, V_2 as in [12], that is

$$U = u\cos\mu, \quad V_1 = (0, \cdots, 0, \cos\psi\sin\mu, 0), \quad V_2 = (0, \cdots, 0, 0, \sin\psi\sin\mu),$$

where $u = (u^1, \ldots, u^6, 0, 0)$ subject to the constraint $(u, u) = 1$ describing the unit round $\mathbf{S}^5$. The deformed norm $\xi^2 \equiv (X, QX)$ then becomes

$$\xi^2 = \eta \cos^2\mu + (\eta_1 \cos^2\psi + \eta_2 \sin^2\psi) \sin^2\mu \tag{5.16}$$

which becomes 1 in the $SO(8)$-invariant limit $\eta = \eta_1 = \eta_2 = 1$ to ensure the correct normalization of the unwarped metric g. The vector u spans the $\mathbf{S}^5$ given by Hopf fibration on $\mathbf{CP}^2$ base. This can be understood by rewriting $(du)^2$ as

$$(du)^2 = (du)^2 - (u, Jdu)^2 + (u, Jdu)^2 \equiv ds^2_{FS(2)} + (u, Jdu)^2,$$

where $ds^2_{FS(2)} \equiv (du)^2 - (u, Jdu)^2$ denotes the Fubini-Study metric on $\mathbf{CP}^2 \cong \mathbf{S}^4$ and (u, Jdu) is the Hopf fiber on it. As mentioned before, the $SU(3)$-invariant deformation must preserve at least the Fubini-Study metric on $\mathbf{CP}^2$. The $U(1)$ symmetry associated with the Hopf fiber (u, Jdu) is a maximal circle of $\mathbf{S}^5 \subset \mathbf{CP}^3$ and is always preserved. The $U(1)$ symmetry preserved in the $SU(3)\times U(1)$-invariant sector but broken in other cases is another $U(1)$ symmetry related to (X, JdX), namely the Hopf fiber on $\mathbf{CP}^3$.

In $SU(3)\times U(1)$ limit, one of the local frames must be fixed to the direction of the Hopf fiber (X, JdX) [12]. Therefore it seems plausible that one of the local frames for the $SU(3)$-invariant 7-manifold, say e^7, is given by

$$e^7 = \xi^{-1}\sqrt{\gamma + 1}\left[(U, JdU) + \sqrt{\frac{\eta_1}{\eta_2}}\,(V_1, JdV_2) + \sqrt{\frac{\eta_2}{\eta_1}}\,(V_2, JdV_1)\right] \tag{5.17}$$

which in fact turns to be $\xi^{-1}\cosh\chi\,(X, JdX)$ in the $SU(3) \times U(1)$ limit $\eta_1 = \eta_2$, $\gamma = \sinh^2\chi$. Then one can rewrite the metric (5.15) such that

$$ds_0^2 = \frac{1}{\eta}\,\cos^2\mu\, ds^2_{FS(2)} + \frac{1}{\xi^2}\,\cos^2\mu \sum_{i,j=1}^{3} M_{ij}\,\omega^i \otimes \omega^j + e^7 \otimes e^7.$$

where $\omega^1 = d\mu$, $\omega^2 = d\psi$, $\omega^3 = (u, Jdu)$ and M_{ij}'s are components of the mixing matrix M given by

$$M = \begin{bmatrix} f_3 & f_0 f_1 & -f_1 \\ f_0 f_1 & f_0 f_2 & -f_2 \\ -f_1 & -f_2 & f_2 f_0^{-1} \end{bmatrix} \tag{5.18}$$

with polynomials

$$\begin{aligned}
f_0 &= \frac{\eta}{\sqrt{\eta_1\eta_2}}, \\
f_1 &= \frac{(\eta_1 - \eta_2)\cos\mu\sin\mu\cos\psi\sin\psi}{\sqrt{\eta_1\eta_2}}, \\
f_2 &= \frac{(\eta_1\cos^2\psi + \eta_2\sin^2\psi)\sin^2\mu}{\sqrt{\eta_1\eta_2}},
\end{aligned}$$

$$f_3 = 2\sin^2\mu + \frac{\eta}{\eta_1\eta_2}\cos^2\mu\,(\eta_1\sin^2\psi + \eta_2\cos^2\psi) + \frac{1}{\eta}\tan^2\mu\sin^2\mu\,(\eta_1\cos^2\psi + \eta_2\sin^2\psi). \tag{5.19}$$

If the seventh frame e^7 we chose is indeed correct, the mixing matrix M must be rank 2 so that its eigenvalues can be $\lambda_+, \lambda_-, 0$ including zero. Equivalently, the bilinear form must be of the form

$$\sum_{i,j=1}^{3} M_{ij}\,\omega^i \otimes \omega^j = \lambda_+\,\omega^+ \otimes \omega^+ + \lambda_-\,\omega^- \otimes \omega^-$$

where ω^+, ω^- are eigenvectors for λ_+, λ_- respectively. This is in fact the case as one can see by diagonalizing M. Two nonzero eigenvalues λ_+, λ_- are solutions to the quadratic equation

$$f(\lambda) \equiv \lambda^2 - \Big[\,f_3 + f_2\,(f_0 + f_0^{-1})\,\Big]\,\lambda - (f_0 + f_0^{-1})\,\Big(f_1^2 f_0 - f_2 f_3\Big) = 0$$

which says that both eigenvalues λ_+ and λ_- are always positive and the corresponding eigenvectors ω_+, ω_- are determined as

$$\omega^{\pm} = \frac{1}{\sqrt{(\lambda_\pm - \lambda_\mp)(f_3 - \lambda_\mp)}}\Big[-(f_3 - \lambda_\mp)\,\omega^1 - f_1 f_0\,\omega^2 + f_1\omega^3\Big].$$

Thus we arrive at the unwarped frames $e^{\,i}$'s given by

$$\begin{aligned}
e^1 &= \eta^{-\frac{1}{2}}\cos\mu\,d\theta, \\
e^2 &= \eta^{-\frac{1}{2}}\cos\mu\,\tfrac{1}{2}\sin\theta\,\sigma_1, \\
e^3 &= \eta^{-\frac{1}{2}}\cos\mu\,\tfrac{1}{2}\sin\theta\,\sigma_2, \\
e^4 &= \eta^{-\frac{1}{2}}\cos\mu\,\tfrac{1}{2}\sin\theta\cos\theta\,\sigma_3, \\
e^5 &= \xi^{-1}\cos\mu\sqrt{\frac{\lambda_+}{(\lambda_+ - \lambda_-)(f_3 - \lambda_-)}}\Big[\,f_1(u, Jdu) - f_1 f_0\,d\psi - (f_3 - \lambda_-)\,d\mu\,\Big], \\
e^6 &= \xi^{-1}\cos\mu\sqrt{\frac{\lambda_-}{(\lambda_- - \lambda_+)(f_3 - \lambda_+)}}\Big[\,f_1(u, Jdu) - f_1 f_0\,d\psi - (f_3 - \lambda_+)\,d\mu\,\Big], \\
e^7 &= \xi^{-1}\sqrt{\gamma + 1}\,\Big[\cos^2\mu\,(u, Jdu) + f_2\,d\psi + f_1\,d\mu\,\Big],
\end{aligned} \tag{5.20}$$

where the first four frames are those on $\mathbf{S}^4 \cong \mathbf{CP}^2$ and e^5, e^6 are oriented to ω^+, ω^- respectively.[6]

Now we can calculate the warp factor Δ by using the local frames (5.20). Some straightforward calculations show

$$\bigwedge_{i=1}^{7} e^{\,i} = \sqrt{\frac{\gamma + 1}{\eta^6\eta_1\eta_2}}\,\xi\,\Omega_7 = \frac{c^{\frac{7}{2}}}{a^{\frac{9}{2}} b^{\frac{9}{2}} d^{\frac{5}{2}}}\,\xi\,\Omega_7 \tag{5.21}$$

[6]Note that the warped frames $(E^1, E^2, E^3, E^4, E^5, E^6, E^7)$ here are denoted by $(e^6, e^7, e^8, e^9, e^5, e^{10}, e^{11})$ in Eq. (4.23) in [12].

where Ω_7 is identified with

$$\Omega_7 = \frac{1}{16}\sin(2\mu)\cos^4\!\mu\,\sin^3\!\theta\cos\theta\,d\theta\wedge\sigma_1\wedge\sigma_2\wedge\sigma_3\wedge d\mu\wedge d\psi\wedge(u,Jdu).$$

This identification is correct since the prefactor in Eq. (5.21) becomes 1 in the maximally symmetric $SO(8)$-invariant limit $a=b=c=d=1$. By using Eq. (5.21) in the self-consistent equation (5.14), the warp factor is determined as

$$\Delta = \left(\frac{ab}{cd}\right)^{\frac{1}{6}} c^{-1}\xi^{-\frac{4}{3}}. \tag{5.22}$$

By using this, ξ in Eq. (5.11) can be solved for Δ to yield

$$G = \frac{1}{\Delta\,\xi^2 N}\,g = \sqrt{\Delta}\left(\frac{a^{\frac{5}{4}}b^{\frac{5}{4}}d^{\frac{3}{4}}}{c^{\frac{3}{4}}}\right)L^2 g,$$

which determines the warping of compact 7-dimensional space. Finally the warped line element $ds_7^2 \equiv (dX, G\,dX)$ is determined as

$$ds_7^2 \equiv G_{mn}\,dy^m dy^n = \sqrt{\Delta}\left(\frac{a^{\frac{5}{4}}b^{\frac{5}{4}}d^{\frac{3}{4}}}{c^{\frac{3}{4}}}\right)L^2\sum_{i=1}^{7} e^{\,i}\otimes e^{\,i}, \tag{5.23}$$

where substitution of the local frames $e^{\,i}$ given in Eq. (5.20) produces the warped metric G_{mn} described by the 7-dimensional global coordinates y^m.

5.4 The compact 7-manifold metric for various critical points in $SU(3)$-invariant sector

There must be some comments on the local frames (5.20) here. First, e^5, e^6 given above are not well-defined in $SU(3)\times U(1)$ limit. We have $f_1=0$ in this limit so that the matrix M shows the mixing between ω_1 and ω_2 only. Diagonalizing the matrix M provides two independent frames

$$\rho^{-2}\xi\,d\mu \quad \text{and} \quad \xi^{-1}\omega \equiv \frac{1}{2}\,\xi^{-1}\sin(2\mu)\left(\rho^4(u,Jdu) - \rho^{-4}d\psi\right)$$

as shown in [12]. However, since either of $f_3-\lambda_\pm$ as well as f_1 becomes 0 in this limit, some of the coefficients in e^5, e^6 become indefinite if we naively take the limit. The careful analysis shows that

$$(e^5, e^6) \longrightarrow \begin{cases} (-\rho^{-2}\xi\,d\mu,\ \xi^{-1}\omega) & \text{for} \quad f_3 > f_2\,(f_0+f_0^{-1}) \\ (\xi^{-1}\omega,\ +\rho^{-2}\xi\,d\mu) & \text{for} \quad f_3 < f_2\,(f_0+f_0^{-1}) \end{cases}.$$

Because of this switching, e^5 and e^6 do not have definite physical meaning in $SU(3)\times U(1)$ limit although they are well-defined in generic $SU(3)$-invariant cases.

Secondly, the set of local frames (5.20) is not unique and any orthogonal transformation on three frames e^5, e^6 and e^7 is possible to produce the other frames $\tilde{e}^5$, $\tilde{e}^6$ and $\tilde{e}^7$ satisfying

$$\tilde{e}^5 \otimes \tilde{e}^5 + \tilde{e}^6 \otimes \tilde{e}^6 + \tilde{e}^7 \otimes \tilde{e}^7 \equiv e^5 \otimes e^5 + e^6 \otimes e^6 + e^7 \otimes e^7.$$

In G_2 limit, we have $\eta = \eta_2 = b^2$, $\eta_1 = a^2$ and $\gamma = 0$ in Eq. (5.15) so that ds_0^2 has no contribution of the last term. To preserve the Fubini-Study metric on the common $\mathbf{CP}^2$ as well as the ellipsoidal deformation along V_1 direction, one can choose $\tilde{e}^5$ and $\tilde{e}^7$ so as to be fixed to (u, Jdu) and the seventh component of dV_1, respectively. The remaining frame $\tilde{e}^6$ is determined by completing squares in ds_0^2. Then one can immediately see that $\tilde{e}^5$, $\tilde{e}^6$, $\tilde{e}^7$ are identified respectively with

$$\frac{1}{b}\cos\mu\,(u, Jdu), \qquad \frac{\sin\psi\, d\mu + \sin\mu\cos\mu\cos\psi\, d\psi}{b\sqrt{1-\sin^2\mu\cos^2\psi}}, \qquad \frac{\xi\, d(\cos\psi\sin\mu)}{ab\sqrt{1-\sin^2\mu\cos^2\psi}}. \tag{5.24}$$

However, even if we take G_2 limit in the local frames (5.20), e^5, e^6, e^7 there cannot reproduce the above three frames without recombining into new frames via some orthogonal transformation. In fact, when we derived Eq. (5.20) we picked up the broken Hopf fiber (5.17) as one of the local frames, expecting the restoration of the $U(1)$ symmetry along the Hopf fiber in $SU(3) \times U(1)$ limit. However in G_2 limit there is no restoration of the $U(1)$ symmetry and we have no reason to choose the broken Hopf fiber as one of the frames. We will discuss more on this shortly.

Let us look at the consistency of our results in both $SU(3) \times U(1)$ and G_2-invariant sectors by reconstructing the 7-manifold metric from the local frames (5.20).

- $SU(3)\times U(1)$-invariant sector: In $SU(3)\times U(1)$ limit, we have $b = 1/a$ and $d = c = \cosh\chi$ so that Eqs. (5.22) and (5.23) reproduce the correct warping of the 7-dimensional metric in section 2 [12]:

$$\Delta = (\xi\cosh\chi)^{-\frac{4}{3}}, \quad ds_7^2 = \sqrt{\Delta}\, L^2 ds_0^2 \quad \text{with} \quad \xi^2 = \rho^{-2}\cos^2\mu + \rho^6\sin^2\mu.$$

In this limit, we have $\eta = \rho^{-2}$, $\eta_1 = \eta_2 = \rho^6$ and e^5, e^6 become

$$\rho^{-2}\xi\, d\mu, \quad \text{and} \quad \xi^{-1}\cos\mu\sin\mu\left(\rho^4(u, Jdu) - \rho^{-4}d\psi\right)$$

as mentioned before. The first four frames are those for $\mathbf{CP}^2$ and $e^7 = \xi^{-1}\cosh\chi\,(X, JdX)$ with $\chi \neq 0$ shows the stretching of Hopf fiber on $\mathbf{CP}^3$. Combining the first six frames provides

$$\rho^4 \sum_{i=1}^{6} e^i \otimes e^i \;=\; \xi^2 d\mu^2 + \cos^2\mu\left[\rho^6 ds^2_{FS(2)} + \xi^{-2}\sin^2\mu\left(\rho^6(u, Jdu) - \rho^{-2}d\psi\right)^2\right],$$

which is nothing but the ellipsoidal deformation of the Fubini-Study metric on $\mathbf{CP}^3$ (See Eq. (B.6) in appendix B). Therefore the compact 7-manifold is given by the ellipsoidal deformation of stretched $\mathbf{S}^7$ as shown in [12].

There is an $\mathcal{N}=2$ supersymmetric critical point specified by $\rho = 3^{1/8}$, $\cosh(2\chi)=2$. This is the $SU(3)\times U(1)$-invariant critical point found in [18]. The $\mathcal{N}=2$ RG flow in section 2 carries this critical point to the $SO(8)$-invariant critical point specified by $\rho=1$, $\chi=0$ [9, 12]. The $SU(4)^-$-invariant critical point in [17] is obtained by further taking the limit $\rho=1$ in $SU(3)\times U(1)$-invariant sector. Since we have $\xi=1$, the warp factor has no dependence on μ and is just a scaling factor in the sense of 7 dimensions. The compact 7-manifold is a stretched $\mathbf{S}^7$ and its $\mathbf{CP}^3$ base is described by the Fubini-Study metric given in Eq. (B.6). The $SU(4)^-$ critical point is non-supersymmetric and there is no RG flow carrying it to the $SO(8)$ critical point [7].

• G_2-invariant sector: To look at this sector, it is better to use the other global coordinates given in appendix C. It is obtained by doing the replacement

$$\cos\mu = \sin\theta_6\sin\theta_7, \quad \sin\mu\cos\psi = \cos\theta_7, \quad \text{and} \quad \sin\mu\sin\psi = \cos\theta_6\sin\theta_7,$$

with $\phi+\psi=\theta_5$ in the previous coordinates of Hopf fibration on $\mathbf{CP}^3$. The deformed norm (5.16) is now rewritten as

$$\xi^2 = \eta_1\cos^2\theta_7 + (\eta\sin^2\theta_6 + \eta_2\cos^2\theta_6)\sin^2\theta_7.$$

In G_2 limit, we have $c=a$, $d=b$ so that Eqs. (5.22) and (5.23) reproduce the correct warping of the 7-dimensional metric in section 3 [2, 10]:

$$\Delta = a^{-1}\xi^{-\frac{4}{3}}, \quad ds_7^2 = \sqrt{\Delta\, a}\, b^2L^2ds_0^2, \quad \text{with} \quad \xi^2 = a^2\cos^2\theta_7 + b^2\sin^2\theta_7.$$

Note that $\eta=\eta_2=b^2$, $\eta_1=a^2$ and $\gamma=0$ in this limit and there is no stretching of $\mathbf{S}^7$. As mentioned before, it is better to transform the last three of the generic frames (5.20) into $\tilde{e}^5$, $\tilde{e}^6$, $\tilde{e}^7$ in Eq. (5.24). They are now simply given by

$$\tilde{e}^5 = b^{-1}\sin\theta_7\sin\theta_6\,(u, Jdu), \quad \tilde{e}^6 = b^{-1}\sin\theta_7\, d\theta_6, \quad \tilde{e}^7 = (ab)^{-1}\xi\, d\theta_7,$$

and are subject to the identity $\sum_{i=5}^7 \tilde{e}^i\otimes\tilde{e}^i \equiv \sum_{i=5}^7 e^i\otimes e^i$. Thus the unwarped metric for the compact 7-manifold is obtained as

$$\sum_{i=1}^7 e^i\otimes e^i = \frac{1}{a^2b^2}\,\xi^2 d\theta_7^2 + \frac{1}{b^2}\sin^2\theta_7\Big[d\theta_6^2 + \sin^2\theta_6\,\left(ds^2_{FS(2)} + (u, Jdu)^2\right)\Big],$$

which describes the ellipsoidally deformed $\mathbf{S}^7$ [2]. Note that the inside of the square bracket is the metric on $\mathbf{S}^6 \cong G_2/SU(3)$ preserving the Fubini-Study metric on $\mathbf{CP}^2$.

There is an $\mathcal{N}=1$ supersymmetric critical point specified by $a=\sqrt{\frac{6\sqrt{3}}{5}}$ and $b=\sqrt{\frac{2\sqrt{3}}{5}}$. This is the G_2-invariant critical point found in [2]. The $\mathcal{N}=1$ RG flow in section 3 carries this critical point to the $SO(8)$-invariant critical point specified by $a=b=1$ [10]. The $SO(7)^+$-invariant critical point corresponds to $a=1/b=5^{1/4}$ showing the ellipsoidal deformation of the 7-manifold. The difference from the G_2 critical point is that there is no field strength of 11-dimensional gauge field in $SO(7)^+$ [19]. Although this critical point is non-supersymmetric, there exists an RG flow connecting it to the $SO(8)$ critical point [7]. The $SO(7)^-$-invariant critical point corresponds to $a = b = \frac{\sqrt{5}}{2}$ so that $\xi = \frac{\sqrt{5}}{2}$ showing no deformation of $\mathbf{S}^7$. The compact 7-manifold is in fact the parallelized round $\mathbf{S}^7$ characterized by the nontrivial field strength of 11-dimensional gauge field [20]. The $SO(7)^-$ critical point is non-supersymmetric and has no RG flow carrying it to the $SO(8)$ critical point.

6 Discussions

In this section, we will discuss our obtained results (5.20), (5.22) and (5.23) in the point of view of 11-dimensional supergravity. First of all, by looking at the deformed norm ξ in Eq. (5.16) we notice that coordinate dependence of the warp factor is not so simple for G_2-invariant sector. We derived the local frames (5.20) such that the restoration of $U(1)$ symmetry associated with the Hopf fiber on $\mathbf{CP}^3$ is obvious in $SU(3) \times U(1)$ limit. However, such an $U(1)$ symmetry does not exist in G_2 limit and the local frames are not appropriate to look at the ellipsoidal deformation of the G_2-invariant 7-manifold.

Therefore, the global coordinates appropriate to describe the compact 7-manifold crucially depends on the base 6-sphere which is $\mathbf{CP}^3$ for the $SU(3) \times U(1)$-invariant sector, whereas $G_2/SU(3)$ for the G_2-invariant sector. The base manifold $\mathbf{CP}^3 \cong \mathbf{S}^6$ is nothing but the homogeneous space $SU(4)/[SU(3) \times U(1)]$ characterized by the Kähler form J [17]. Both $SU(4)^-$ and $SU(3) \times U(1)$-invariant 7-manifolds share the same $U(1)$ symmetry along the Hopf fiber (X, JdX) so that Hopf fibration on $\mathbf{CP}^3$ is useful in those cases. On the other hand, the 7-manifolds with at least G_2 invariance, namely $SO(7)^\pm$ and G_2-invariant 7-manifolds, share the 6-sphere given by $G_2/SU(3)$ described as an $\mathbf{S}^6$ embedded in $\mathbf{R}^7$ spanned by imaginary octonions [21]. Since the ellipsoidal deformation is transverse to this $\mathbf{S}^6$, it is well described by using the 7-dimensional coordinates in appendix C. Therefore, the complication in the warp factor and in the local frames is due to the difference in $\mathbf{S}^6$ base between the two sectors.

However, in spite of the difference in the $\mathbf{S}^6$ base, both sectors share the same $\mathbf{S}^4 \cong \mathbf{CP}^2$. This $\mathbf{S}^4$ is obvious in $SU(3) \times U(1)$-invariant sector, while it is implicit in the 6-sphere of $G_2/SU(3)$ in G_2-invariant sector. It was pointed out in [12, 13] that replacing the $\mathbf{S}^4$ with $\mathbf{S}^2 \times \mathbf{S}^2$ provides another 11-dimensional solution corresponding to the $d = 4$, $\mathcal{N} = 2$, $SU(3) \times U(1)$-invariant RG flow. It may be interesting to look at whether such a replacement yields another 11-dimensional solution corresponding to the $d = 4$, $\mathcal{N} = 1$, G_2-invariant RG flow.

As summarized in section 5, the $SU(3)$-invariant sector contains various critical points. It is still unclear why some of critical points have holographic RG flows to the $SO(8)$ critical point but is not so for others. To answer this question, we have to solve the 11-dimensional Einstein-

Maxwell equations to complete the 11-dimensional lift of whole $SU(3)$-invariant sector including RG flows. The 11-dimensional metric is given by Eq. (1.1) where the compact 7-manifold metric G_{mn} and the warp factor Δ are completely determined by Eqs. (5.23), (5.22) in the local frames (5.20). The geometric parameters a, b, c, d depend on the AdS_4 radial coordinate r and are subject to the RG flow equations (4.2) in 4-dimensional gauged supergravity [7]. The local frame (5.20) found in this paper will be useful to achieve this work. For example, as performed in [12], one can easily make an ansatz for the 3-form gauge field by using the local frames. We postpone this subject for future study.

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Appendix A The 7-dimensional inverse metric encoded in $SU(3)$-singlet vevs of $d = 4$, $\mathcal{N} = 8$ gauged supergravity

According to the formula (5.1) of 7-dimensional inverse metric, one gets all the elements of $\Delta^{-1}G^{AB}$ as follows. For simplicity, we did not write them completely but some of them can be read off from the known expressions. For example, $\Delta^{-1}G^{13}$ can be written as $\Delta^{-1}G^{12}$ by replacing x_2 with x_3 and x_7 with $-x_6$. We list them below. For simplicity, $L^{-2}\Delta^{-1}G^{AB}$ is denoted by (AB).

$$
\begin{aligned}
(11) &= \frac{1}{2}(ac^2+bcd)x_2^2+\frac{1}{2}(ac^2+bcd)x_3^2+\frac{1}{4}(a^3+2a^2b+ab^2+ac^2-2acd+ad^2)x_4^2 \\
&\quad +ac^2x_5^2+\frac{1}{2}(ac^2+bcd)x_6^2+\frac{1}{2}(ac^2+bcd)x_7^2+\frac{1}{2}(a^3-ab^2-ac^2+ad^2)x_4x_8 \\
&\quad +\frac{1}{4}(a^3-2a^2b+ab^2+ac^2+2acd+ad^2)x_8^2, \\
(12) &= \frac{1}{2}(-ac^2-bcd)x_1x_2+\frac{1}{2}(-ac^2+bcd)x_2x_5+\frac{1}{2}(a^2b+ab^2-acd-bcd)x_4x_7 \\
&\quad +\frac{1}{2}(a^2b-ab^2-acd+bcd)x_7x_8, \\
(13) &= \frac{1}{2}(-ac^2-bcd)x_1x_3+\frac{1}{2}(-ac^2+bcd)x_3x_5+\frac{1}{2}(-a^2b-ab^2+acd+bcd)x_4x_6 \\
&\quad +\frac{1}{2}(-a^2b+ab^2+acd-bcd)x_6x_8,
\end{aligned}
$$

$$
\begin{aligned}
&= (12) \text{ with } (x_2 \to x_3, x_7 \to -x_6),\\
(14) &= -\frac{1}{4}a(a^2+2ab+b^2+c^2-2cd+d^2)x_1x_4+\frac{1}{4}a(a^2-b^2+c^2-d^2)x_4x_5\\
&\quad +\frac{1}{4}a(a^2-b^2-c^2+d^2)x_1x_8-\frac{1}{4}a(a^2-2ab+b^2-c^2+2cd-d^2)x_5x_8,\\
(15) &= \frac{1}{2}(ac^2-bcd)x_2^2+(ac^2-bcd)x_3^2+\frac{1}{4}(a^3-ab^2+ac^2-ad^2)x_4^2-ac^2x_1x_5\\
&\quad +\frac{1}{2}(ac^2-bcd)x_6^2+\frac{1}{2}(ac^2-bcd)x_7^2+\frac{1}{2}(a^3+ab^2-ac^2-ad^2)x_4x_8\\
&\quad +\frac{1}{4}(a^3-ab^2+ac^2-ad^2)x_8^2,\\
(16) &= (12) \text{ with } (x_7 \to x_3, x_2 \to x_6),\\
(17) &= (12) \text{ with } (x_2 \to x_7, x_7 \to -x_2),\\
(18) &= \frac{1}{4}a(-a^2+b^2+c^2-d^2)x_1x_4+\frac{1}{4}a(-a^2-2ab-b^2+c^2+2cd+d^2)x_4x_5\\
&\quad +\frac{1}{4}a(-a^2+2ab-b^2-c^2-2cd-d^2)x_1x_8+\frac{1}{4}a(-a^2+b^2-c^2+d^2)x_5x_8,\\
(22) &= \frac{1}{2}(ac^2+bcd)x_1^2+bcdx_3^2+\frac{1}{2}(bcd+ad^2)x_4^2+\frac{1}{2}(ac^2+bcd)x_5^2+(ac^2-bcd)x_1x_5\\
&\quad +bcdx_6^2+ab^2x_7^2+(-bcd+ad^2)x_4x_8+\frac{1}{2}(bcd+ad^2)x_8^2,\\
(23) &= -bcdx_2x_3+(-ab^2+bcd)x_6x_7,\\
(24) &= \frac{1}{2}(-bcd-ad^2)x_2x_4+\frac{1}{2}(-a^2b-ab^2+acd+bcd)x_1x_7+\frac{1}{2}(bcd-ad^2)x_2x_8\\
&\quad +\frac{1}{2}(-a^2b+ab^2+acd-bcd)x_5x_7,\\
(25) &= (12) \text{ with } (x_5 \leftrightarrow x_1, x_4 \leftrightarrow x_8),\\
(26) &= (23) \text{ with } (x_3 \to x_6, x_6 \to -x_3),\\
(27) &= -ab^2x_2x_7,\\
(28) &= (24) \text{ with } (x_4 \leftrightarrow x_8, x_1 \leftrightarrow x_5),\\
(33) &= (22) \text{ with } (x_2 \leftrightarrow x_3, x_7 \leftrightarrow x_6),\\
(34) &= (24) \text{ with } (x_2 \to x_3, x_7 \to -x_6),\\
(35) &= (12) \text{ with } (x_5 \leftrightarrow x_1, x_4 \leftrightarrow x_8, x_2 \to x_3, x_7 \to -x_6),\\
(36) &= -ab^2x_3x_6,\\
(37) &= (23) \text{ with } (x_2 \to x_7, x_7 \to -x_2),\\
(38) &= (24) \text{ with } (x_4 \leftrightarrow x_8, x_1 \leftrightarrow x_5, x_2 \to x_3, x_7 \leftrightarrow -x_6),\\
(44) &= \frac{1}{4}(a^3+2a^2b+ab^2+ac^2-2acd+ad^2)x_1^2+\frac{1}{2}(bcd+ad^2)x_2^2+\frac{1}{2}(bcd+ad^2)x_3^2\\
&\quad +\frac{1}{2}(a^3-ab^2+ac^2-ad^2)x_1x_5+\frac{1}{4}(a^3-2a^2b+ab^2+ac^2+2acd+ad^2)x_5^2
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2}(bcd+ad^2)x_6^2+\frac{1}{2}(bcd+ad^2)x_7^2+ad^2x_8^2, \\
(45) &= (18) \ \text{with} \ (x_4 \leftrightarrow x_8, x_1 \leftrightarrow x_5), \\
(46) &= (24) \ \text{with} \ (x_2 \to x_6, x_7 \to x_3), \\
(47) &= (24) \ \text{with} \ (x_2 \to x_7, x_7 \to -x_2), \\
(48) &= \frac{1}{4}(a^3-ab^2-ac^2+ad^2)x_1^2+\frac{1}{2}(-bcd+ad^2)x_2^2+\frac{1}{2}(-bcd+ad^2)x_3^2 \\
& +\frac{1}{2}(a^3+ab^2-ac^2-ad^2)x_1x_5+\frac{1}{4}(a^3-ab^2-ac^2+ad^2)x_5^2 \\
& +\frac{1}{2}(-bcd+ad^2)x_6^2+\frac{1}{2}(-bcd+ad^2)x_7^2-ad^2x_4x_8, \\
(55) &= (11) \ \text{with} \ (x_5 \leftrightarrow x_1, x_4 \leftrightarrow x_8), \\
(56) &= (12) \ \text{with} \ (x_4 \leftrightarrow x_8, x_1 \leftrightarrow x_5, x_2 \leftrightarrow x_6, x_7 \to x_3), \\
(57) &= (12) \ \text{with} \ (x_2 \to x_7, x_5 \leftrightarrow x_1, x_4 \leftrightarrow x_8, x_7 \to -x_2), \\
(58) &= (14) \ \text{with} \ (x_1 \leftrightarrow x_5, x_4 \leftrightarrow x_8), \\
(66) &= (22) \ \text{with} \ (x_6 \to x_2, x_7 \leftrightarrow x_3), \\
(67) &= (-ab^2+acd)x_2x_3-bcdx_6x_7, \\
(68) &= (24) \ \text{with} \ (x_1 \leftrightarrow x_5, x_4 \leftrightarrow x_8, x_2 \to x_6, x_7 \to x_3), \\
(77) &= (22) \ \text{with} \ (x_7 \to x_2), \\
(78) &= (24) \ \text{with} \ (x_4 \leftrightarrow x_8, x_1 \leftrightarrow x_5, x_2 \to x_7, x_7 \to -x_2), \\
(88) &= (44) \ \text{with} \ (x_5 \leftrightarrow x_1, x_4 \leftrightarrow x_8).
\end{aligned}
$$

In the $SU(4)^-$-invariant limit ($a=b=1, d=c$) the above generated results can be arranged into the 8×8 matrix

$$\Delta^{-1}G^{-1} = c^2(I-xx^T)+(1-c^2)\,(\tilde{J}\,x)(\tilde{J}\,x)^T$$

by introducing the Kähler form $\tilde{J}$ given by

$$\tilde{J} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.1}$$

This Kähler form is transformed into the standard one J given by Eq. (5.2) in the text via

$\tilde{J} = R^T J R$ with the orthogonal matrix

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{A.2}$$

Appendix B The global coordinates for $\mathbf{S}^7$ as Hopf fibration on $\mathbf{CP}^3$

In this appendix, we summarize basic properties of Hopf fibration and Fubini-Study metric on $\mathbf{CP}^n$ $(n = 2, 3)$ used in the text.

- $\mathbf{R}^8$ embedding of Hopf fibration on $\mathbf{CP}^3$:

$$X = u\cos\mu + v\sin\mu \quad \text{with} \quad u = (u^1, \dots, u^6, 0, 0), \quad v = (0, \dots, 0, v^7, v^8).$$

$$\begin{aligned} u^1 + iu^2 &= \sin\theta\cos(\tfrac{1}{2}\alpha_1)\, e^{\frac{i}{2}(\alpha_2+\alpha_3)} e^{i(\phi+\psi)}, \\ u^3 + iu^4 &= \sin\theta\sin(\tfrac{1}{2}\alpha_1)\, e^{-\frac{i}{2}(\alpha_2-\alpha_3)} e^{i(\phi+\psi)}, \\ u^5 + iu^6 &= \cos\theta\, e^{i(\phi+\psi)}, \\ v^7 + iv^8 &= e^{i\psi}. \end{aligned} \tag{B.1}$$

- Imaginary quaternion basis:

$$\begin{aligned} \sigma_1 &= \cos\alpha_3\, d\alpha_1 + \sin\alpha_1 \sin\alpha_3\, d\alpha_2, \\ \sigma_2 &= \sin\alpha_3\, d\alpha_1 - \sin\alpha_1 \cos\alpha_3\, d\alpha_2, \\ \sigma_3 &= d\alpha_3 + \cos\alpha_1\, d\alpha_2, \end{aligned} \tag{B.2}$$

satisfying the Maurer-Cartan equation:

$$d\sigma_i = \frac{1}{2}\epsilon_{ijk}\,\sigma_j \wedge \sigma_k.$$

- Hopf fiber on $\mathbf{CP}^2$:

$$(u, Jdu) = d(\phi+\psi) + \frac{1}{2}\sin^2\theta\,\sigma_3. \tag{B.3}$$

- Fubini-Study metric on $\mathbf{CP}^2$:

$$ds^2_{FS(2)} \equiv (du)^2 - (u, Jdu)^2 = d\theta^2 + \frac{1}{4}\sin^2\theta\,(\sigma_1^2 + \sigma_2^2 + \cos^2\theta\,\sigma_3^2). \tag{B.4}$$

• Hopf fiber on $\mathbf{CP}^3$:

$$(X, JdX) = \sin^2\mu \, d\psi + \cos^2\mu \, (u, Jdu) = d\psi + \cos^2\mu \left(d\phi + \frac{1}{2} \sin^2\theta \, \sigma_3 \right). \tag{B.5}$$

• Fubini-Study metric on $\mathbf{CP}^3$:

$$ds^2_{FS(3)} \equiv (dX)^2 - (X, JdX)^2 = d\mu^2 + \cos^2\mu \left[ds^2_{FS(2)} + \sin^2\mu \left(d\phi + \frac{1}{2} \sin^2\theta \, \sigma_3 \right)^2 \right]. \tag{B.6}$$

To get understanding of Hopf fibration, let us demonstrate to reconstruct the compact 7-manifold metric for the $SU(4)^-$-invariant critical point. At this critical point, we have $a = b = 1, d = c$ and $\eta = 1 = \eta_1 = \eta_2$. Due to the common singularity in the $SU(3) \times U(1)$-invariant sector, we get either $(e^5, e^6) = (-\, d\mu, \, \omega)$ or $(e^5, e^6) = (\omega, \, +d\mu)$ where we used the fact that $\xi = 1$. Then $e^5 \otimes e^5 + e^6 \otimes e^6$ will lead to

$$d\mu^2 + \sin^2\mu \cos^2\mu \left(d\phi + \frac{1}{2} \sin^2\theta \, \sigma_3 \right)^2 .$$

Combining other $e^i, i = 1, 2, 3, 4$ with e^5, e^6, the metric can be written as

$$\sum_{i=1}^{6} e^i \otimes e^i = d\mu^2 + \cos^2\mu \left[ds^2_{FS(2)} + \sin^2\mu \left(d\phi + \frac{1}{2} \sin^2\theta \, \sigma_3 \right)^2 \right]$$

which is nothing but the standard Fubini-Study metric on $\mathbf{CP}^3$ [17]. Moreover, e^7 in this limit will be $e^7 = c \left(\cos^2\mu \, (u, Jdu) + \sin^2\mu \, d\psi \right)$. Using the explicit form of (u, Jdu) one gets

$$e^7 \otimes e^7 = c^2 \left[d\psi + \cos^2\mu \left(d\phi + \frac{1}{2} \sin^2\theta \, \sigma_3 \right) \right]^2$$

that is equal to the $U(1)$ bundle and ψ is the coordinate on the $U(1)$ fibers. According to the result of [17], the compact 7-manifold metric in the above will be an Einstein which corresponds to the trivial $SO(8)$-invariant metric on the round 7-sphere when $c^2 = 1$. On the other hand, at the $SU(4)^-$-invariant critical point($c = \sqrt{2}$), $U(1)$ fibers over $\mathbf{CP}^3$ are stretched by a factor $\sqrt{2}$ and the compact 7-manifold metric is not Einstein.

Appendix C   The global coordinates for $\mathbf{S}^7$ with $G_2/SU(3)$ base

In this appendix, we describe the 7-dimensional coordinatization appropriate for the base 6-sphere of $G_2/SU(3)$.

• $\mathbf{R}^8$ embedding of $\mathbf{S}^7$ with $G_2/SU(3)$ base:

$$X = u \sin\theta_6 \sin\theta_7 + \tilde{v} \quad \text{with} \quad u = (u^1, \ldots, u^6, 0, 0), \quad \tilde{v} = (0, \ldots, 0, \cos\theta_7, \cos\theta_6 \sin\theta_7),$$

where u is the same as above except for the replacement $\phi+\psi \to \theta_5$.

• Relation to the Hopf fibration on $\mathbf{CP}^3$:

$$\cos\theta_7 = \sin\mu\cos\psi, \quad \cos\theta_6\sin\theta_7 = \sin\mu\sin\psi, \quad \sin\theta_6\sin\theta_7 = \cos\mu, \quad \text{and} \quad \theta_5 = \phi+\psi.$$

$$d\theta_7 = \frac{d(\cos\psi\sin\mu)}{\sqrt{1-\sin^2\mu\cos^2\psi}}, \quad \sin\theta_7\, d\theta_6 = \frac{\sin\psi\, d\mu + \sin\mu\cos\mu\cos\psi\, d\psi}{\sqrt{1-\sin^2\mu\cos^2\psi}}. \tag{C.1}$$

$$\begin{aligned} -\sin\mu\, d\mu &= \cos\theta_6\sin\theta_7\, d\theta_6 + \sin\theta_6\cos\theta_7\, d\theta_7, \\ \sin^2\mu\, d\psi &= \cos\theta_6\, d\theta_7 - \sin\theta_6\cos\theta_7\sin\theta_7\, d\theta_6. \end{aligned} \tag{C.2}$$

• The metric on $\mathbf{S}^6 \cong G_2/SU(3)$:

$$d\Omega_6^2 \equiv d\theta_6^2 + \sin^2\theta_6 \left(ds^2_{FS(2)} + (u, Jdu)^2\right). \tag{C.3}$$

References

[1] B. de Wit and H. Nicolai, Nucl. Phys. **B208** (1982) 323; Phys. Lett. **B108** (1982) 285.

[2] B. de Wit, H. Nicolai and N.P. Warner, Nucl. Phys. **B255** (1985) 29.

[3] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231, `hep-th/9711200`.

[4] E. Witten, Adv. Theor. Math. Phys. **2** (1998) 253, `hep-th/9802150`.

[5] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. **B428** (1998) 105, `hep-th/9802109`.

[6] E. Cremmer and B. Julia, Phys. Lett. **B80** (1978) 48.

[7] C. Ahn and K. Woo, Nucl. Phys. **B599** (2001) 83, `hep-th/0011121`.

[8] N.P. Warner, Phys. Lett. **B128** (1983) 169.

[9] C. Ahn and J. Paeng, Nucl. Phys. **B595** (2001) 119, `hep-th/0008065`.

[10] C. Ahn and T. Itoh, Nucl. Phys. **B627** (2002) 45, `hep-th/0112010`.

[11] C. Ahn and S.-J. Rey, Nucl. Phys. **B572** (2000) 188, `hep-th/9911199`.

[12] R. Corrado, K. Pilch and N.P. Warner, Nucl. Phys. **B629** (2002) 74, `hep-th/0107220`.

[13] C.V. Johnson, K.J. Lovis and D.C. Page, JHEP **0110** (2001) 014, `hep-th/0107261`.

[14] P.G.O. Freund and M.A. Rubin, Phys. Lett. **B97** (1980) 233.

[15] B. de Wit and H. Nicolai, Nucl. Phys. **B281** (1987) 211.

[16] C. Ahn and K. Woo, Nucl. Phys. **B634** (2002) 141, `hep-th/0109010`.

[17] C.N. Pope and N.P. Warner, Phys. Lett. **B150** (1985) 352.

[18] H. Nicolai and N.P. Warner, Nucl. Phys. **B259** (1985) 412.

[19] B. de Wit and H. Nicolai, Phys. Lett. **B148** (1984) 60.

[20] F. Englert, Phys. Lett. **B119** (1982) 339.

[21] M. Günaydin and N.P. Warner, Nucl. Phys. **B248** (1984) 685.